2018

(April)

MATHEMATICS

(Elective/Honours)

(Geometry and Vector Calculus)

(GHS-21)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** from each Unit

UNIT—I

1. *(a)* If by rotation of the rectangular axes the equation $17x^2 + 18xy - 7y^2 = 1$ reduces to the form $ax^2 + by^2 = 1$, find the angle through which the axes are rotated. Also find the values of a and b. 4

(b) Prove that the equation

$$2x^2 + xy - 6y^2 - 6x + 23y - 20 = 0$$

represents a pair of straight lines. Find the coordinates of their point of intersection. 5

(c) Reduce the equation

$$17x^2 + 12xy + 8y^2 - 46x - 28y + 17 = 0$$

to the standard form. 6

2. (a) Prove that the equation of the tangent to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0$$

at the origin is $gx + fy = 0$. 6

(b) Find the diameter of the conic

$$15x^2 - 20xy + 16y^2 = 1$$

conjugate to the diameter $y + 2x = 0$. 5

(c) Prove that the two hyperbolas

$$4x^2 + 3xy + 5x + 21 = 0$$

and $$x^2 - 4xy - 3x + 19 = 0$$

have a common asymptote. Also find the other asymptotes. 4

(Continued)

UNIT—II

3. (a) Show that the locus of the point of intersection of any two perpendicular tangents to the parabola $y^2 = 4ax$ is the directrix. 5

(b) Find the asymptotes of the hyperbola 5

$$xy + ax + by = 0$$

(c) A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find the equation of the tangent and its point of contact. 5

4. (a) Show that the normal to the rectangular hyperbola $xy = c^2$ at the point t meets the curve again at the point t' such that $t^3 t' = -1$. 5

(b) Prove that the straight line $\frac{ax}{3} + \frac{by}{4} = c$ will be a normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

if $5c = a^2 e^2$. 5

(c) Prove that two tangents can be drawn from a given point to an ellipse. 5

(Turn Over)

UNIT—III

5. *(a)* Show that the equation of the plane which passes through the point (2, −3, 8) and is normal to the line joining the points (3, 4, −1) and (2, −1, 5) is $x+5y-6z+61=0$. 5

(b) Find the equation to the locus of the point whose distance from the origin is equal to its perpendicular distance from the plane $2x+3y-6z=9$. 5

(c) Find the shortest distance between the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

and $$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$

Also show that the lines are coplanar. 5

6. *(a)* Prove that the plane

$$2x-2y+z+12=0$$

touches the sphere

$$x^2+y^2+z^2-2x-4y+2z=3$$ 5

(Continued)

(b) Find the equation of the right circular cone, whose vertex is (3, 2, 1), axis is the line

$$\frac{x-3}{4}=\frac{y-2}{1}=\frac{z-1}{3}$$

and semi-vertical angle is 30°. 5

(c) Find the equation of the cylinder generated by the lines parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{5}$, the guiding curve being the conic $x=0,\ y^2=8z$. 5

UNIT—IV

7. *(a)* If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$, show that the vectors $\vec{a}, \vec{b}, \vec{c}$ are orthogonal in pairs and $|\vec{b}|=1$, $|\vec{c}|=|\vec{a}|$. 5

(b) Show that the four points $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar if

$$[\vec{b}\ \vec{c}\ \vec{d}]+[\vec{c}\ \vec{a}\ \vec{d}]+[\vec{a}\ \vec{b}\ \vec{d}]=[\vec{a}\ \vec{b}\ \vec{c}]$$ 5

(c) Show that the volume of the parallelopiped whose edges are represented by $(3i+2j-4k)$, $(3i+j+3k)$ and $(i-2j+k)$ is 49 cubic units. 5

(Turn Over)

8. *(a)* Show that a necessary and sufficient condition for a vector $\vec{u}(t)$ to have a constant direction is $\vec{u} \times \frac{d\vec{u}}{dt} = 0$. 6

(b) Find the unit tangent vector at any point on the curve $x = a\cos t$, $y = a\sin t$, $z = bt$. 4

(c) If $\vec{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$, then find

$$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}$$

5

UNIT—V

9. *(a)* Find the directional derivative of

$$\phi = (x^2 + y^2 + z^2)^{-1/2}$$

at the point $P(3, 1, 2)$ in the direction of the vector $yz\hat{i} + zx\hat{j} + xy\hat{k}$. 5

(b) Show that

$$\text{grad}\, e^{(x^2 + y^2 + z^2)} = 2e^{r^2}$$

where $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. 5

(c) If

$$\vec{A} = (x + y + 1)\hat{i} + \hat{j} + (-x - y)\hat{k}$$

prove that $\vec{A} \cdot (\nabla \times \vec{A}) = 0$. 5

(Continued)

10. *(a)* Prove that

$$\text{div}\, \hat{r} = \frac{2}{r}$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

5

(b) Find the equations of the tangent plane and normal line to the surface

$$f = 2xz^2 - 3xy - 4x - 7$$

at the point $(1, -1, 2)$. 6

(c) Find the gradient and unit vector normal to the surface $f = x^2 + y - z$ at the point $(1, 0, 0)$. 4

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